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Wasserstein Contractive Estimates for the Homogeneous Landau Equation

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Contents

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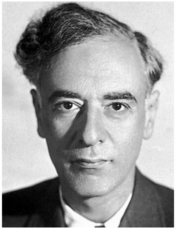
$$\partial_t f + v \cdot \nabla_x f = q(f), \quad f = f(x, v, t).$$

The **collision operator** q acts on v, is non-linear, nonlocal, and second order "elliptic":



Lev. Landau

$$q(f) = \text{div}_v \int_{\mathbb{R}^3} |v - w|^{\gamma+2} \mathbb{P}(v - w)(\nabla_v - \nabla_w)(f(v)f(w))\, dw,$$

where $\mathbb{P}(z) = \mathbb{I} - \frac{z \otimes z}{|z|^2}$ is the projection to $z^{\perp}$.

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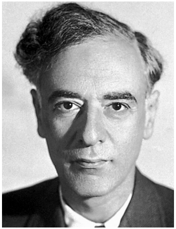

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The collision operator

The collision operator can be written as

$$\begin{aligned} q(f) &= \mathrm{div}_v \int_{\mathbb{R}^3} |v-w|^{\gamma+2}\mathbb{P}(v-w)(\nabla_v - \nabla_w)(f(v)f(w))\,dw \\ &= \mathrm{div}_v\left[A_\gamma[f](v)\nabla_v f(v) - \mathrm{div}_v\left(A_\gamma[f](v)\right)f(v)\right] \\ &= \mathrm{tr}\left(A_\gamma[f]D_v^2 f\right) + c_\gamma f(v)\int_{\mathbb{R}^3} f(w)|v-w|^\gamma\,dw, \end{aligned}$$

where

$$A_\gamma[f](v) = \int_{\mathbb{R}^3} |v-w|^{2+\gamma}\,\mathbb{P}(v-w)f(w)\,dw.$$

As $\gamma \to -3$, one obtains that in the **Coulomb case**

$$q(f) = \mathrm{tr}\left(A[f]D^2 f\right) + 8\pi f^2.$$

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In the **space homogeneous case** we have

$$f(x,v,t) = f(v,t), \quad \partial_t f = q(f).$$

Some of the most important facts:

- The only steady states are **Maxwellians** of the form

$$m(v) = ae^{-b|v-u|^2}, \quad a,b > 0, u \in \mathbb{R}^d.$$

- Moments of order 0, 1, 2 are constant:

$$\frac{d}{dt}\int f\,dv = \frac{d}{dt}\int fv\,dv = \frac{d}{dt}\int f|v|^2\,dv = 0.$$

  If $\int f_{\text{in}}(1,v,|v|^2)dv = (1,0,3)$ then $m(v) = (2\pi)^{-\frac{3}{2}}e^{-\frac{|v|^2}{2}}$.

- Known **Lyapunov functionals**

$$\frac{d}{dt}\int f\log f\,dv \le 0, \quad \frac{d}{dt}\int \frac{|\nabla f|^2}{f} \le 0.$$

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Existing results

State of the field for the homogeneous Landau equation:

- **Existence of weak solutions** (known): Ukai, DiPerna, Lions, Villani, Desvilletes, ...
- **Asymptotic behavior** (known): Desvillettes, Villani, Carrapatoso, ...
- **Existence of smooth solutions** (known): Desvilletes, Villani, Guillen, Silvestre, Ji, Gualdani, Golding, Loher, ...
- **Uniqueness** (mostly known): Open for L^1 initial data. Various results by Fournier, Guerin, Gualdani, Golding, Loher, ...

General global well-posedness remains an open problem in the inhomogeneous setting.

Blow-up was only ruled out recently. Note that for $\gamma = -3$,

$$\partial_t f = \text{tr}\left(A[f] D^2 f\right) + 8\pi f^2,\ A[f] = \int_{\mathbb{R}^d} |v-w|^{-1}\,\mathbb{P}(v-w) f(w)\,dw.$$

Theorem (Guillen, Silvestre, 2023)

Let f be a classical solution to the Landau equation for any $|\gamma| \le \sqrt{19}$. Then the Fisher information $i(f)$ is monotone decreasing:

$$i(f) = \int \frac{|\nabla f|^2}{f}\,dv.$$

For example, note that

$$i(f) = 4\int |\nabla \sqrt{f}|^2\,dv = 4\|\sqrt{f}\|_{\dot{H}^1} \gtrsim \|\sqrt{f}\|_{L^6} = \|f\|_{L^3}.$$

New Lyapunov functional

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Other functionals under control?

Objective: Finding other monotone or bounded functionals.

Motivation: Consider $\gamma = 0$ and f radial. Then

$$\partial_t f = \mathrm{div}(\nabla f(v) + vf) = \mathrm{div}(m\nabla(f/m)),$$

where $m(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{1}{2}|v|^2}$.

In this case the following functionals are decreasing:

- Entropy and Fisher information.
- **Relative L^p norms** $\int \left(\frac{f}{m} - 1\right)^p m\,dv$.
- 2-**Wasserstein distance** between any pair of solutions.

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Optimal transport metric

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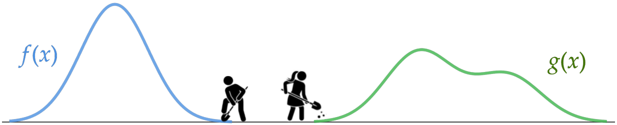

Definition

Given two probability densities $f, g : \mathbb{R}^d \longrightarrow \mathbb{R}$, we define the 2-**Wasserstein distance** as

$$d_2^2(f, g) = \inf_{T_\# f = g} \int |T(x) - x|^2 f(x)\, dx,$$

where $T_\# f$ denotes the push-forward measure.

The topology is slightly stronger than weak convergence:

$$\lim_n d_2(f_n, f) = 0 \iff f_n \rightharpoonup f, \quad \lim_n \int |x|^2 f_n(x)\, dx = \int |x|^2 f(x)\, dx.$$

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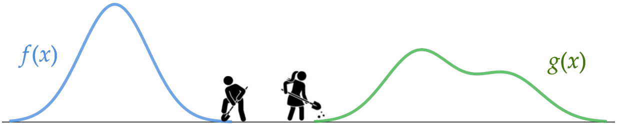

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Alternative expression of d_2^2

The metric satisfies the **Benamou-Brenier** formula:

$$d_2^2(f,g) = \inf \int_0^1 \int_{\mathbb{R}^d} |\vec{v}(x,s)|^2 \rho(x,s)\, dx\, ds,$$

where the infimum is taken over solutions of

$$\partial_s \rho + \mathrm{div}\,(\vec{v}\rho) = 0, \qquad \rho(\cdot,0) = f,\ \rho(\cdot,1) = g.$$

At the optimum, we have $\vec{v}(x,s) = \nabla u(x,s)$ and

$$\partial_s u + \frac{1}{2}|\nabla u|^2 = 0, \quad d_2^2(f,g) = \int_{\mathbb{R}^d} u(\cdot,1)g - \int_{\mathbb{R}^d} u(\cdot,0)f,$$

which gives the alternative formulation

$$\boxed{d_2^2(f,g) = \sup\left\{ \int_{\mathbb{R}^d} u(\cdot,1)g - \int_{\mathbb{R}^d} u(\cdot,0)f \,:\, \partial_s u + \frac{1}{2}|\nabla u|^2 = 0 \right\}.}$$

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Maxwell molecules case

Theorem

Let f, g be smooth solutions of the Landau equation for $\gamma = 0$ with mass 1. Then $d_2(f, g)$ is decreasing in t.

This has been proven by Fournier using different methods.

Key idea of Guillen-Silvestre. The Landau operator is the projection of a "lifted" operator Q on $\mathbb{R}^6$.

$$Q(F) = (\text{div}_v - \text{div}_w)\Big[|v-w|^{2+\gamma}\mathbb{P}(v-w)(\nabla_v F - \nabla_w F)\Big].$$

Then the idea is to compare the evolution of

$$\begin{cases} \partial_t f = q(f) \\ f_0 = f_{\text{in}} \end{cases}, \qquad \begin{cases} \partial_t F = Q(F) \\ F_0 = f_{\text{in}} \otimes f_{\text{in}} \end{cases}$$

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Lifting functionals

Proposition (Lifting)

Let $i(f), I(F)$ be functionals such that

1. $I(f \otimes f) = 2i(f)$.
2. *If* $F(v,w) = F(w,v)$ *then* $I(F) \geq 2i(\pi_1 f)$ *where* $\pi_1 f = \int_{\mathbb{R}^3} F \, dw$ *is the marginal of* F.

If f, F *are solutions of the previous equations then*

$$\frac{d}{dt}\bigg|_{t=0} i(f) \leq \frac{1}{2} \frac{d}{dt}\bigg|_{t=0} I(F).$$

- Entropy, fisher information and d_2^2 lift!
- Other functionals like L^p norms don't lift.
- These properties holds for any γ.

Lifting functionals

The optimal vector field to compute $d_2^2(f \otimes f, g \otimes g)$ is form

$$u(v,w,s) = u(v,s) + u(w,s), \quad d_2^2(f,g) = \int u(\cdot,1)g - \int u(\cdot,0)f.$$

From here the lifting follows easily:

$$\begin{aligned}
\frac{d}{dt}[d_2^2(f,g)] &= \int u(v,1)q(g) - \int u(v,0)q(f) \\
&= \iint u(v,1)Q(g \otimes g) - \iint u(v,0)Q(f \otimes f) \\
&= \frac{1}{2}\iint u(v,w,1)Q(g \otimes g) - \frac{1}{2}\iint u(v,w,0)Q(f \otimes f) \\
&= \frac{1}{2}\frac{d}{dt}[d_2^2(F,G)].
\end{aligned}$$

The problem is reduced to the linear equation $\partial_t F = Q(F)$!

Notation: Given a functional $I(F)$, we denote

$$\langle I'(F), \xi \rangle := \frac{d}{ds}\Big|_{s=0} I(F+s\xi), \quad \langle I''(F)\xi, \xi \rangle := \frac{d^2}{ds^2}\Big|_{s=0} I(F+s\xi).$$

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Then, given solutions of $\partial_t F = \Delta F$, $\partial_t G = \Delta G$, we have

$$\begin{aligned}\frac{d}{dt}[d_2^2(F,G)] &= \sum_k \Big\langle (d_2^2)'(F,G), (\partial_{x_k x_k} F, \partial_{x_k x_k} G) \Big\rangle \\ &= -\sum_k \Big\langle (d_2^2)''(F,G)\big(\partial_{x_k} F, \partial_{x_k} G\big), (\partial_{x_k} F, \partial_{x_k} G) \Big\rangle \leq 0,\end{aligned}$$

where we used that d_2^2 is convex.

Key idea: Decompose Q into directions where we have diffusion. Following Guillen-Silvestre, we define

$$b_1(z) := \begin{pmatrix} 0 \\ -z_3 \\ z_2 \end{pmatrix},\ b_2(z) := \begin{pmatrix} z_3 \\ 0 \\ -z_1 \end{pmatrix},\ b_3(z) := \begin{pmatrix} -z_2 \\ z_1 \\ 0 \end{pmatrix},\ \tilde{b}_k = \begin{pmatrix} b_k \\ -b_k \end{pmatrix}.$$

These satisfy:

- $\mathrm{span}(b_k(z)) = \langle z \rangle^\perp$, • $\operatorname{div} b_k = 0$, • $\sum_k (b_k \otimes b_k) = |z|^2 \mathbb{P}(z)$

Setting $\tilde{b}_k = \tilde{b}_k(v-w)$, we have

$$Q(F) = \sum_k \operatorname{div}\left[|v-w|^\gamma (\tilde{b}_k \otimes \tilde{b}_k) \nabla F\right] = |v-w|^\gamma \sum_k \left(\tilde{b}_k \cdot \nabla \left(\tilde{b}_k \cdot \nabla F\right)\right)$$

Contractivity for $\gamma = 0$

For the Maxwell molecules case $\gamma = 0$, we have

$$Q(F) = \sum_k \left(\tilde{b}_k \cdot \nabla \left(\tilde{b}_k \cdot \nabla F \right) \right).$$

Then setting $L_k(F) := \tilde{b}_k \cdot \nabla F$, we have

$$\begin{aligned} \frac{d}{dt}\left[d_2^2(F,G) \right] &= \sum_k \left\langle (d_2^2)'(F,G), ((L_k \circ L_k)(F), (L_k \circ L_k)(G)) \right\rangle \\ &= -\sum_k \left\langle (d_2^2)''(F,G) \left(L_k F, L_k G \right), \left(L_k F, L_k G \right) \right\rangle \leq 0. \end{aligned}$$

Key property and why this does not work for $\gamma \neq 0$:

The flow of $\begin{pmatrix} \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} b_k(v-w) \\ -b_k(v-w) \end{pmatrix}$ generates an isometry.

Case of general potential

Theorem (Non Coulomb)

Let f, g solve the Landau equation for $-3 < \gamma \leq 0$. If $f, g \in L^1_t L^p_v$ for $p > \frac{3}{3+\gamma}$ then there is $C_{\gamma,p} > 0$ such that

$$d_2^2(f_t, g_t) \leq d_2^2(f_{\text{in}}, g_{\text{in}}) \exp\left(C_{\gamma,p} \int_0^t \left(\|f_s\|_{L^p(\mathbb{R}^3)} + \|g_s\|_{L^p(\mathbb{R}^3)}\right) ds\right).$$

These estimates were found by Fourier and Guerín in 2008.

Theorem (Maxwell molecules)

If $\gamma = 0$ then $d_2^2(f, g)$ is decreasing with dissipation

$$\frac{d}{dt}[d_2^2(f, g)] = -\sum_k \int_0^1 \iint_{\mathbb{R}^6} |\nabla(\tilde{b}_k \cdot \nabla u)|^2 \rho \, dv dw \, ds,$$

where ρ interpolates $\rho(\cdot, 0) = f \otimes f$, $\rho(\cdot, 0) = g \otimes g$ and

$$\partial_s u + \frac{1}{2}|\nabla u|^2 = 0, \quad \partial_s \rho + \operatorname{div}(\rho \nabla u) = 0.$$

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Theorem (Coulomb)

Let f, g solve the Landau equation for $\gamma = -3$. If $f, g \in L^1_t L^\infty_v$ then

$$\frac{d}{dt}[d_2^2(f,g)] \lesssim \left(\|f\|_{L^\infty(\mathbb{R}^3)} + \|g\|_{L^\infty(\mathbb{R}^3)}\right) d_2^2(f,g) |\log d_2^2(f,g)|.$$

which implies the estimate

$$d_2^2(f_t, g_t) \leq [d_2^2(f_{\text{in}}, g_{\text{in}})]^{e^{-\alpha(t)}}, \ \alpha(t) = C\int_0^t \left(\|f_s\|_{L^\infty(\mathbb{R}^3)} + \|g_s\|_{L^\infty(\mathbb{R}^3)}\right) ds.$$

- Fournier proved an equivalent result in 2009 using stochastic analysis.
- This estimate implies uniqueness within the class $L^1_t L^\infty_v$.

Overview of the proof for $\gamma \neq 0$

- As before, we compute $\frac{d}{dt}[d_2^2(F,G)]$ where F, G solve

$$\begin{cases} \partial_t F = Q(F) \\ F_0 = f_{\text{in}} \otimes f_{\text{in}} \end{cases}, \quad \begin{cases} \partial_t G = Q(G) \\ G_0 = g_{\text{in}} \otimes g_{\text{in}}. \end{cases}$$

- We use the heat equation as inspiration.

If F, G solve $\partial_t F = \Delta F$, $\partial_t G = \Delta G$ then

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Example (Heat equation)

If $\partial_t f = \Delta f$, $\partial_t g = \Delta g$ have mass 1 then

$$\frac{d}{dt}\left[d_2^2(f,g)\right] = -\int_0^1 \int_{\mathbb{R}^d} \left|D^2 u\right|^2 \rho(x)\, dx\, ds$$

where $\rho(x,s)$, $u(x,s)$, $0 \leq s \leq 1$ satisfy

$$\partial_s \rho + \operatorname{div}(\rho \nabla u) = 0, \qquad \rho_0 = f,\ \rho_1 = g, \qquad \partial_s u + \frac{1}{2}|\nabla u|^2 = 0.$$

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Step 1: Imitate computation for the heat equation.

Denoting $F = f \otimes f$, $G = g \otimes g$, we have

$$\frac{d}{dt}\left[d_2^2(F,G)\right] = \iint u_1 Q(G) - \iint u_0 Q(F) = \int_0^1 \frac{d}{ds} \iint Q(u)\rho.$$

Now the expression for Q in terms of $\tilde{b}_k$ gives

$$= \sum_k \iint \left[\nabla\left(\tilde{b}_k \cdot \nabla\left(\tilde{b}_k \cdot \nabla u\right)\right) \cdot \nabla u - \frac{1}{2}\tilde{b}_k \cdot \nabla\left(\tilde{b}_k \cdot \nabla |\nabla u|^2\right)\right] r^\gamma \rho$$
$$+ \sum_k \gamma \iint \left(\tilde{b}_k \cdot \nabla\left(\tilde{b}_k \cdot \nabla u\right)\right)(\hat{n} \cdot \nabla u)\, r^{\gamma-2}\rho,$$

where $r = |v - w|$ and $\hat{n} = \begin{pmatrix} v - w \\ w - v \end{pmatrix}$.

The final expression is

$$\frac{d}{dt}\left[d_2^2(f,g)\right] = -\sum_{k=1}^{3}\int_0^1 \iint \left|\nabla\left(\tilde{b}_k\cdot\nabla u\right)\right|^2 r^\gamma \rho + \sum_{k=1}^{3}\gamma\int_0^1 \iint \left(\tilde{b}_k\cdot\nabla\left(\tilde{b}_k\cdot\nabla u\right)\right)(\hat{n}\cdot\nabla u)\, r^{\gamma-2}\rho.$$

where $r = |v-w|$ and $\hat{n} = \begin{pmatrix} v-w \\ w-v \end{pmatrix}$. Then we decompose

$$\left|\nabla\left(\tilde{b}_k\cdot\nabla u\right)\right|^2 r^\gamma = \frac{1}{2}\left(\tilde{b}_k\cdot\nabla\left(\tilde{b}_k\cdot\nabla u\right)\right)^2 r^{\gamma-2} + \cdots$$

which gives

$$\boxed{\frac{d}{dt}\left[d_2^2(f,g)\right] \le \frac{3\gamma^2}{2}\int_0^1 \iint (\hat{n}\cdot\nabla u)^2 r^{\gamma-2}\rho}$$

Estimate for $-3 < \gamma < 0$

Using $u(v,w) = u(v) + u(w)$, $\rho(v,w) = \rho(v)\rho(w)$ gives

- $$\begin{aligned} &\leq \int_0^1 \iint_{\mathbb{R}^6} |\nabla u(v) - \nabla u(w)|^2 |v-w|^\gamma \rho(v)\rho(w)\, dv\, dw\, ds \\ &\leq 2\int_0^1 \iint_{\mathbb{R}^6} |v-w|^\gamma\, |\nabla u(v)|^2\, \rho(v)\rho(w)\, dv\, dw\, ds \\ &= 2\int_0^1 \int_{\mathbb{R}^3} |\nabla u(v)|^2\, \rho(v) \Bigg(\underbrace{\int_{\mathbb{R}^3} \rho(w)\, |v-w|^\gamma\, dw}_{:=(I_\gamma \rho)(v)} \Bigg) dv\, ds \\ &\leq 2\, \|I_\gamma \rho\|_{L^\infty(\mathbb{R}^3\times[0,1])}\, d_2^2(f,g). \end{aligned}$$

Then, given $p > \frac{3}{3+\gamma}$, we have

$$\|I_\gamma \rho\|_{L^\infty(\mathbb{R}^3\times[0,1])} \leq C_{\gamma,p} \left(\|f\|_{L^p(\mathbb{R}^3)} + \|g\|_{L^p(\mathbb{R}^3)} \right).$$

Estimate for $-3 < \gamma < 0$

Using $u(v,w) = u(v) + u(w)$, $\rho(v,w) = \rho(v)\rho(w)$ gives

$$
\begin{aligned}
&\le \int_0^1 \iint_{\mathbb{R}^6} |\nabla u(v) - \nabla u(w)|^2 |v-w|^\gamma \rho(v)\rho(w)\, dv\, dw\, ds \\
&\le 2 \int_0^1 \iint_{\mathbb{R}^6} |v-w|^\gamma \, |\nabla u(v)|^2 \, \rho(v)\rho(w)\, dv\, dw\, ds \\
&= 2 \int_0^1 \int_{\mathbb{R}^3} |\nabla u(v)|^2 \, \rho(v) \Bigg(\underbrace{\int_{\mathbb{R}^3} \rho(w) \, |v-w|^\gamma \, dw}_{:=(I_\gamma \rho)(v)} \Bigg) dv\, ds \\
&\le 2 \, \|I_\gamma \rho\|_{L^\infty(\mathbb{R}^3 \times [0,1])} \, d_2^2(f,g).
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$$

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$$

Estimate for $\gamma = -3$

For $\gamma = -3$ we use

$$-\frac{1}{1-s}\mathbb{I} \leq D^2 u \leq \frac{1}{s}\mathbb{I} \implies |\nabla u(v) - \nabla u(w)| \leq \frac{|v-w|}{\min(s, 1-s)}.$$

Then, given $\varepsilon > 0$ small, and denoting $\theta_\varepsilon(s) = [\min(s, 1-s)]^{-2\varepsilon}$

$$\begin{aligned}\frac{d}{dt}\left[d_2^2(f,g)\right] &\lesssim \int_0^1 \theta_\varepsilon(s) \iint_{\mathbb{R}^6} |v-w|^{-3+2\varepsilon} |\nabla u(v) - \nabla u(w)|^{2(1-\varepsilon)} \rho(v)\rho(w)\, dv\, dw\, ds. \\ &\lesssim \left(\|f\|_{L^\infty(\mathbb{R}^3)} + \|g\|_{L^\infty(\mathbb{R}^3)}\right) \varepsilon^{-1} \left[d_2^2(f,g)\right]^{1-\varepsilon}.\end{aligned}$$

Choosing $\varepsilon = -\frac{1}{\log d_2^2(f,g)}$ for small $d_2^2(f,g)$ gives an estimate

$$\frac{d}{dt}\left[d_2^2(f,g)\right] \lesssim \left(\|f\|_{L^\infty(\mathbb{R}^3)} + \|g\|_{L^\infty(\mathbb{R}^3)}\right) d_2^2(f,g) |\log d_2^2(f,g)|.$$

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What have we achieved?

- New non-stochastic proofs of Fournier's contractive estimates in the 2-Wasserstein distance.
- Dissipation expression for $\gamma = 0$.
- Entropic optimal transport is decreasing for $\gamma = 0$.
- Explicit expression of $\frac{d}{dt}[d_2^2(f,g)]$.

Open problems:

- Is $d_2^2(f,g)$ decreasing for any range of $\gamma \neq 0$?
- Obtain similar estimates in the inhomogeneous setting.

We consider f a smooth solution of the homogeneous Landau equation for Maxwell molecules ($\gamma = 0$) and assume

$$\int_{\mathbb{R}^3} f_0(1, v, |v|^2)\, dv = (1, 0, 3), \qquad m(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{1}{2}|v|^2}.$$

Theorem (Maxwell Molecules)

The relative L^2 norm of f with respect to m decays exponentially:

$$\int_{\mathbb{R}^3} \left(\frac{f_t}{m} - 1\right)^2 m\, dv \leq C e^{-4t} \int_{\mathbb{R}^3} \left(\frac{f_{\text{in}}}{m} - 1\right)^2 m\, dv, \quad C \leq e^{\frac{1}{6}}.$$

Moreover, it is monotonically decreasing for $t \geq t_0$,

$$t_0 = \max\left\{0, \frac{1}{8} \log\left(|\mathbb{M}_{f_{\text{in}}} - \mathbb{I}|_2\right)\right\}, \quad (\mathbb{M}_{f_{\text{in}}})_{ij} = \int_{\mathbb{R}^3} f_{\text{in}} v_i v_j\, dv.$$

Step 1: Reduction to a linear equation.

$$A[f_t] = \int \mathbb{P}(v-w)|v-w|^2 f_t(w)\,dw = |v|^2\mathbb{P}(v) + 3\mathbb{I} - \mathbb{M}_{f_t}$$

where $\mathbb{M}_{f_t} = \int f_t v_i v_j \, dv$. Then, one can show

$$\frac{d}{dt}\mathbb{M}_{f_t} = -12(\mathbb{M}_{f_t} - \mathbb{I}) \implies \mathbb{M}_{f_t} = \mathbb{I} + e^{-12t}(\mathbb{M}_{f_{\text{in}}} - \mathbb{I}).$$

Rotating the initial data, we reduce the Landau equation to

$$\partial_t f = 3\,\text{div}(m\nabla(f/m)) + \text{div}(|v|^2\mathbb{P}(v)\nabla f) - \text{div}(fv + \mathbb{M}_{f_t}\nabla f),$$

where

$$\mathbb{M}_{f_t} = \text{diag}(\lambda_i(t)), \quad \lambda_i(t) = \int f_t v_i^2\,dv = 1 + e^{-12t}(\lambda_i(0) - 1).$$

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Step 2: Write as perturbation of Fokker-Planck.

We use the test function f/m on

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$$\frac{1}{2}\frac{d}{dt}\int\left(\frac{f}{m}-1\right)^2 m \leq -2\int\left|\nabla\left(\frac{f}{m}\right)\right|^2 m + \sum_i(\lambda_i(t)-1)\int\left(\left[\partial_{v_i}\left(\frac{f}{m}\right)\right]^2 - \frac{1}{2}\frac{f^2}{m^2}v_i^2\right)m.$$

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Step 3: Control lower order term using Poincaré inequality.

Lemma

For any smooth function $\varphi : \mathbb{R}^d \to \mathbb{R}$ we have

$$\int |x_i|^2 \varphi^2(x) m(x)\, dx \leq 2 \int |\partial_{x_i} \varphi|^2 m(x)\, dx + 2 \int \varphi^2(x) m(x)\, dx.$$

Moreover, if φ has zero average $\int \varphi(x) m(x)\, dx = 0$ then

$$\int \varphi^2(x) m(x)\, dx \leq \int |\nabla \varphi|^2 m(x)\, dx.$$

Then, denoting $E(t) = \int \left(\frac{f}{m} - 1 \right)^2 m$, we obtain the inequality

$$E'(t) \leq -4(1 - Ce^{-12t}) E(t), \quad C = C\left(\lambda_i(0)\right).$$

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