Uniqueness of bounded solutions for the fuzzy Landau equation

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Motivation

We study uniqueness and stability of the **fuzzy Landau equation**:

$$\partial_t f + v\cdot\nabla_x f = \nabla_v \cdot \int_{\mathbb{R}^6} \kappa(x-x_*)\Phi(v-v_*)\nabla_{v-v_*}\big(f(x,v)f(x_*,v_*)\big)\,dx_*\,dv_*$$

where $\kappa \geq 0$, $\Phi(z) = |z|^{2+\gamma}\mathbb{P}(z)$, $-3\leq\gamma\leq 1$, and $\mathbb{P}(z) = \mathrm{Id} - \frac{z\otimes z}{|z|^2}$

- It may be viewed as a space delocalization of the classical Landau equation, which is a fundamental kinetic model in plasma physics.
- Setting $\kappa(x) = \delta_0(x)$, we recover the classical Landau equation.
- The most physically relevant case corresponds to the Coulomb interaction with $\gamma = -3$.

Main result

Consider two weak solutions f, g of fuzzy Landau for $-3\leq\gamma\leq -2$ and $\sqrt{\kappa}$ Lipschitz. If $f, g \in L^1_t L^p_v L^1_x$ for $p = \frac{3}{3+\gamma} \in [3,\infty]$ then

$$\sup_{0\leq t\leq T} d_2^2\big(f(t), g(t)\big) \leq \omega\left(d_2^2\big(f(0), g(0)\big)\right),$$

where the modulus ω depends on $\|f+g\|_{L^1_t L^p_v L^1_x}$, and $\|\sqrt{\kappa}\|_{\mathrm{Lip}}$.

Definitions

Wasserstein distance between two probability measures on $\mathbb{R}^d$:

$$d_2^2(f,g) = \inf_{\Pi\in\Gamma(f,g)} \int_{\mathbb{R}^{2d}} |x-y|^2\,d\Pi(x,y),$$

where $\Gamma(f,g)$ denotes probability measures on $\mathbb{R}^{2d}$ with marginals f, g.

Modulus of continuity is a continuous, nondecreasing function $\omega : [0,\infty) \to [0,\infty)$ such that $\omega(0) = 0$.

Weak solution f of the fuzzy Landau equation is such that

$$\frac{d}{dt}\int_{\mathbb{R}^6} \varphi f = \int_{\mathbb{R}^6} \left[-(v\cdot\nabla_x\varphi) + 2(\kappa b * f)\cdot\nabla_v\varphi + (\kappa\Phi * f) : D_v^2\varphi\right] f,$$

for all $\varphi \in \mathcal{C}_b^2(\mathbb{R}^6)$, where $b(z) := \mathrm{div}_z\Phi(z) = -2|z|^\gamma z$.

General setup

We extend the techniques of [1, 2] to equations depending on $x\in\mathbb{R}^n$

$$\partial_t f = \nabla\cdot\Big(cf + (b*f)f\Big) + D^2 : \Big((A*f)f\Big), \tag{1}$$

where $D : A = \sum_{ij}\partial_{ij}A^{ij}$, and $c, b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, satisfy

(A1) The matrix A can be expressed as $A = \sigma\sigma^T$.

(A2) c is globally Lipschitz.

(A3) x decomposes as $x = (x^1,\dots,x^k,x') \in (\mathbb{R}^d)^k\times\mathbb{R}^{n-dk}$ so that

$$|b(x)| \leq K\left(1 + \sum_{i=1}^k |x^i|^{\gamma+1}\right), \quad |\sigma(x)| \leq K\left(1+\sum_{i=1}^k |x^i|^{\frac{\gamma}{2}+1}\right).$$

(A4) b, σ satisfy a local Lipschitz estimate away from $x^i = 0$:

$$\frac{|b(x)-b(y)|}{|x-y|} + \frac{|\sigma(x)-\sigma(y)|^2}{|x-y|^2} \leq K\left(1+\sum_{i=1}^k \left(|x^i|^\gamma + |y^i|^\gamma\right)\right),$$

Given solutions f, g of (1), we generate $\Pi_t \in \Gamma(f(t), g(t))$ and estimate

$$d_2^2\big(f(t), g(t)\big) \leq \int_{\mathbb{R}^{2n}} |x-y|^2\,d\Pi_t(x,y).$$

Choice of coupling evolution

We generate Π_t by requiring that it satisfies the parabolic equation

$$\partial_t\Pi = \mathrm{div}_{x,y}\left(\left(\bar{c} + (\bar{b}*\Pi^*)\right)\Pi\right) + D^2_{x,y} : \left(\left((\overline{A}*\Pi^*)\right)\Pi\right), \tag{2}$$

where $\Pi_t^* \in \Gamma(f(t), g(t))$ denotes the optimal coupling, $\Pi_0 = \Pi_0$, and

$$\bar{c} = \begin{pmatrix} c(x)\\ c(y)\end{pmatrix}, \quad \bar{b} = \begin{pmatrix} b(x)\\ b(y)\end{pmatrix}, \qquad \overline{A} = \begin{pmatrix} \sigma(x)\sigma^T(x) & \sigma(x)\sigma^T(y)\\ \sigma(y)\sigma^T(x) & \sigma(y)\sigma^T(y)\end{pmatrix}.$$

This equation is carefully chosen so ensure the marginals

$$\mu_t = \int \Pi_t\,dy, \qquad \nu := \int \Pi_t\,dx$$

remain equal to $f(t)$, $g(t)$, respectively. Integrating (2) in y yields

$$\partial_t\mu = \nabla\cdot(c\mu + (b*f)\mu) + D^2 : ((A*f)\mu).$$

Therefore, f_t, μ_t solve the same **linear equation**, which implies $\mu = f$. The existence of such a Π_t is proved using a coupled system of SDEs.

Main computation

Simple integration by parts gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int |x-y|^2\,d\Pi(x,y) &= \int (c(x)-c(y))\cdot(x-y)\,d\Pi(x,y)\\ &+ \int (b(x-x_*) - b(y-y_*))\cdot(x-y)\,d\Pi^*(x_*,y_*)\,d\Pi(x,y)\\ &+ \int |\sigma(x-x_*) - \sigma(y-y_*)|^2\,d\Pi^*(x_*,y_*)\,d\Pi(x,y).\end{aligned}$$

This yields uniqueness if all coefficients are Lipschitz. However, the singularity in (A4) creates **logarithmically** diverging integrals

$$\int_{|x-x_*|\geq\varepsilon} |x^i - x^i_*|^\gamma f_t^{(i)}(x_*) \lesssim |\log\varepsilon|\,\|f_t^{(i)}\|_{L^p}, \quad p = \frac{3}{3+\gamma}, \tag{3}$$

where $f_t^{(i)}$ denotes the x^i **marginal** of f_t. We divide into two regions and estimate separately:

- Over $R_1 = \left\{|x^i - x^i_*| \geq \varepsilon, |y^i - y^i_*| \geq\varepsilon, \forall i\right\}$, we use estimate (3).
- Over $R_2 = \mathbb{R}^{4n}\setminus R_1$ we use (A3).

All together:

$$\frac{d}{dt}\int |x-y|^2\,d\Pi_t \lesssim \max_i(\|f_t^{(i)}\|_{L^p}, \|g_t^{(i)}\|_{L^p})\left[\varepsilon + |\log\varepsilon|\int |x-y|^2\,d\Pi_t\right].$$

Setting $\varepsilon = \min\left(\int|x-y|^2\,d\Pi_t, e^{-1}\right)$ yields

$$\frac{d}{dt}\int |x-y|^2\,d\Pi_t \lesssim \max_{1\leq i\leq k}(\|f_t^{(i)}\|_{L^\infty}, \|g_t^{(i)}\|_{L^\infty})\Psi\left(\int |x-y|^2\,d\Pi_t\right),$$

with $\Psi(s) = s(1+|\log s|)$, which implies the stability estimate.

The modulus of continuity ω can be computed by solving a differential equation of the form $\dot{y} = y(1-\log y)$ for $y(0)\ll 1$.

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References

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