F. Unai Caja Lopez

---

# Uniqueness of solutions to the fuzzy Landau equation

---

Universidad Autónoma de Madrid

April 23, 2026

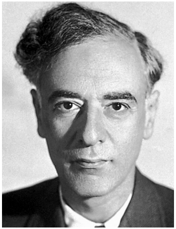
Joint work with M. Delgadino, M. Gualdani and M. Taskovic

# The Landau equation

The **Landau equation** has the form

$$\partial_t f + v \cdot \nabla_x f = q(f), \quad f = f(x, v, t).$$

The **collision operator** $q$ acts on $v$, is non-linear, nonlocal, and second order "elliptic":



Lev. Landau

$$q(f) = \mathrm{div}_v \int_{\mathbb{R}^3} \Phi(v - v_*)(\nabla_v - \nabla_{v_*})\Big(f(x, v) f(x, v_*)\Big)\, dv_*,$$

where $\Phi(z) = |z|^{-1}\mathbb{P}(z)$, $\mathbb{P}(z) = \mathrm{Id}_3 - \frac{z \otimes z}{|z|^2}$.

- Appeared in 1936 as a version of the Boltzmann equation for Coulomb forces
- It as also a correction to the Vlasov Poisson equation.

# Basic properties

The equation may be re-written as

$$\partial_t f(x,v) + v \cdot \nabla_x f(x,v) = \sum_{i,j} a^{ij}[f]\partial_{v_i v_j} f + 8\pi f^2,$$

where

$$\left(a^{ij}[f](x,v)\right)_{ij} = \int_{\mathbb{R}^3} \Phi(v - v_*) f(v_*, x)\, dv_* = (f *_v \Phi)(x,v).$$

**Conserved quantities**:

$$\frac{d}{dt}\int f\, dx\, dv = \frac{d}{dt}\int f v^i\, dx\, dv = \frac{d}{dt}\int f|v|^2\, dx\, dv = 0.$$

**H-theorem**:

$$\frac{d}{dt}\int f \log f\, dv \leq 0.$$

# Basic properties

The equation may be re-written as

$$\partial_t f(x,v) + v \cdot \nabla_x f(x,v) = \sum_{i,j} a^{ij}[f]\partial_{v_i v_j} f + 8\pi f^2,$$

where

$$\left(a^{ij}[f](x,v)\right)_{ij} = \int_{\mathbb{R}^3} \Phi(v-v_*) f(v_*,x)\, dv_* = (f *_v \Phi)(x,v).$$

**Conserved quantities**:

$$\frac{d}{dt}\int f\, dx\, dv = \frac{d}{dt}\int f v^i\, dx\, dv = \frac{d}{dt}\int f|v|^2\, dx\, dv = 0.$$

**H-theorem**:

$$\frac{d}{dt}\int f \log f\, dv \leq 0.$$

# State of the literature

**Space homogeneous:** $f(x,v) = f(v)$

- Existence of $H$-solutions (Villani).
- Global existence of smooth solutions is known for $L^1$ initial data (Guillen, Silvestre, Ji, Gualdani, Golding, Loher, $\ldots$).
- Uniqueness is mostly known. Currently open for $L^1$ initial data (Fournier, Guerin, Gualdani, Golding, Loher, $\ldots$).
- Asymptotic behavior for smooth solutions: Desvillettes, Villani, Carrapatoso.

# State of the literature

**Space inhomogeneous:**

- Asymptotic behavior for smooth solutions: Desvillettes, Villani, Carrapatoso.
- Global existence only known for:
  - Renormalized solutions (DiPerna, Lions).
  - Near equilibrium (Guo, Carrapatoso, Mischler).
  - Near vacuum for moderately soft and hard potentials (Luk, Sanchit)
- Finite time singularities for very hard potentials (Bedrossian, Chen, Gualdani, Ji, Vicol, Yang).

# Homogeneous equation

$$\partial_t f(v) = \int_{\mathbb{R}^3} \underbrace{\mathrm{div}_{v-v_*}\left[\Phi(v-v_*)\nabla_{v-v_*}(f(v)f(v_*))\right]}_{=:Q(f\otimes f)} dv_*.$$

Blow-up was only ruled out recently by

**Theorem (Guillen, Silvestre, 2023)**

*If $f$ is a classical solution of homogeneous Landau then*

$$\frac{d}{dt} i(f) = \frac{d}{dt}\int \frac{|\nabla f|^2}{f}\, dv \leq 0.$$

Key observation:

$$\frac{d}{dt} i(f) = \frac{d}{d\tau}\bigg|_{\tau=0} I(F), \qquad \begin{cases} \partial_\tau F = Q(F), \\ F|_{\tau=0} = f_t \otimes f_t. \end{cases}$$

# Homogeneous equation

$$\partial_t f(v) = \int_{\mathbb{R}^3} \underbrace{\mathrm{div}_{v-v_*}\Big[\Phi(v-v_*)\nabla_{v-v_*}\big(f(v)f(v_*)\big)\Big]}_{=:Q(f\otimes f)}\,dv_*.$$

Blow-up was only ruled out recently by

**Theorem (Guillen, Silvestre, 2023)**

*If $f$ is a classical solution of homogeneous Landau then*

$$\frac{d}{dt} i(f) = \frac{d}{dt}\int \frac{|\nabla f|^2}{f}\,dv \le 0.$$

Key observation:

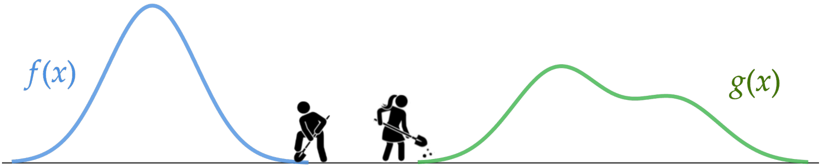
$$\frac{d}{dt} i(f) = \frac{d}{d\tau}\Big|_{\tau=0} I(F), \qquad \begin{cases} \partial_\tau F = Q(F), \\ F|_{\tau=0} = f_t \otimes f_t. \end{cases}$$

# Other functionals under control?

The **2-Wasserstein distance** is

$$d_2^2(f,g) = \inf_{\Pi \in \Gamma(f,g)} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v-w|^2 \, d\Pi(v,w).$$

where $\Gamma(f,g)$ is the subset of $\mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3)$ with marginals $f$, $g$.



If $f, g$ are solutions of homogeneous Landau, then

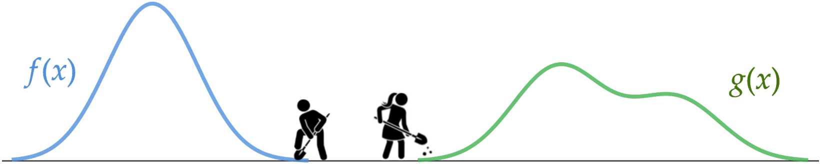
$$\frac{d}{dt} d_2^2(f,g) = \frac{d}{d\tau}\Big|_{\tau=0} d_2^2(F,G), \quad \begin{cases} \partial_\tau F = Q(F), \\ F|_{\tau=0} = f_t \otimes f_t, \end{cases} \quad \begin{cases} \partial_\tau G = Q(G), \\ G|_{\tau=0} = g_t \otimes g_t. \end{cases}$$

# Other functionals under control?

The **2-Wasserstein distance** is

$$d_2^2(f,g) = \inf_{\Pi \in \Gamma(f,g)} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v-w|^2 \, d\Pi(v,w).$$

where $\Gamma(f,g)$ is the subset of $\mathcal{P}(\mathbb{R}^3 \times \mathbb{R}^3)$ with marginals $f$, $g$.



If $f, g$ are solutions of homogeneous Landau, then

$$\frac{d}{dt} d_2^2(f,g) = \frac{d}{d\tau}\Big|_{\tau=0} d_2^2(F,G), \quad \begin{cases} \partial_\tau F = Q(F), \\ F|_{\tau=0} = f_t \otimes f_t, \end{cases} \quad \begin{cases} \partial_\tau G = Q(G), \\ G|_{\tau=0} = g_t \otimes g_t. \end{cases}$$

# Strategy

We use the following formulation of the metric:

$$d_2^2(f,g) = \sup\left\{\int_{\mathbb{R}^d} u(\cdot,1)g - \int_{\mathbb{R}^d} u(\cdot,0)f \,:\, \partial_s u + \frac{1}{2}|\nabla u|^2 = 0\right\}.$$

The optimal $u$ provides the geodesic connecting $f$ and $g$:

$$\partial_s\rho_s + \text{div}\left(\nabla u_s\rho_s\right) = 0, \quad \rho_0 = f,\ \rho_1 = g, \quad \int_{\mathbb{R}^d} |\nabla u_s|^2\, d\rho_s = d_2^2(f,g).$$

**Toy model:** if $\partial_t f = \Delta f$, $\partial_t g = \Delta g$ have unit mass

$$\begin{aligned}\frac{d}{dt}d_2^2(f,g) &= \int_{\mathbb{R}^n} u_1\Delta g\,dx - \int_{\mathbb{R}^n} u_0\Delta f\,dx = \int_0^1 \frac{d}{ds}\int_{\mathbb{R}^n} \Delta u\rho\,dx\,ds \\ &= -\int_0^1\int_{\mathbb{R}^n}\left(\frac{1}{2}\Delta\left|\nabla u\right|^2 - \nabla\Delta u\cdot\nabla u\right)\rho\,dx\,ds \\ &= -\frac{1}{2}\int_0^1\int_{\mathbb{R}^n}\left|D^2u\right|^2\rho\,dx\,ds.\end{aligned}$$

# Strategy

We use the following formulation of the metric:

$$d_2^2(f,g) = \sup\left\{\int_{\mathbb{R}^d} u(\cdot,1)g - \int_{\mathbb{R}^d} u(\cdot,0)f : \partial_s u + \frac{1}{2}|\nabla u|^2 = 0\right\}.$$

The optimal $u$ provides the geodesic connecting $f$ and $g$:

$$\partial_s \rho_s + \text{div}\left(\nabla u_s \rho_s\right) = 0, \quad \rho_0 = f,\ \rho_1 = g, \quad \int_{\mathbb{R}^d} |\nabla u_s|^2\, d\rho_s = d_2^2(f,g).$$

**Toy model:** if $\partial_t f = \Delta f$, $\partial_t g = \Delta g$ have unit mass

$$\begin{aligned}\frac{d}{dt} d_2^2(f,g) &= \int_{\mathbb{R}^n} u_1 \Delta g\, dx - \int_{\mathbb{R}^n} u_0 \Delta f\, dx = \int_0^1 \frac{d}{ds}\int_{\mathbb{R}^n} \Delta u \rho\, dx\, ds \\ &= -\int_0^1 \int_{\mathbb{R}^n} \left(\frac{1}{2}\Delta\left|\nabla u\right|^2 - \nabla\Delta u \cdot \nabla u\right)\rho\, dx\, ds \\ &= -\frac{1}{2}\int_0^1\int_{\mathbb{R}^n}\left|D^2 u\right|^2 \rho\, dx\, ds.\end{aligned}$$

# Homogeneous Landau equation

## Theorem (Fournier, 2010)

*Let $f, g$ be weak solution of homogeneous Landau. Then*

$$\frac{d}{dt}[d_2^2(f,g)] \lesssim \left(\|f\|_{L^\infty(\mathbb{R}^3)} + \|g\|_{L^\infty(\mathbb{R}^3)}\right) \Psi\left(d_2^2(f,g)\right),$$

*where* $\Psi(s) = s(1 + |\log s|)$*, which implies for* $d_2^2(f_0, g_0) \ll 1$

$$d_2^2(f_t, g_t) \leq [d_2^2(f_0, g_0)]^{e^{-\alpha(t)}}, \ \alpha(t) = C\int_0^t \left(\|f_s\|_{L^\infty(\mathbb{R}^3)} + \|g_s\|_{L^\infty(\mathbb{R}^3)}\right) ds.$$

- Fournier used stochastic analysis and a coupling argument.
- We recover the same bound using the lifting property.
- Implies uniqueness within the class $L^1_t L^\infty_v$.

# Sketch of the proof

Recreating the computations for the heat equation shows:

$$\frac{d}{dt} d_2^2(f,g) \lesssim \int_0^1 \int |v-v_*|^{-3} |\nabla u(v) - \nabla u(v_*)|^2 \rho(v)\rho(v_*)\, dv\, dv_*\, ds$$

We use

$$|\nabla u(v) - \nabla u(v_*)| \lesssim_s |v-v_*|, \quad 0 < s < 1.$$

Then, for $0 < \varepsilon < 1$

$$\begin{aligned} \frac{d}{dt} d_2^2(f,g) &\lesssim \int |v-v_*|^{-3+2\varepsilon} |\nabla u(v) - \nabla u(v_*)|^{2(1-\varepsilon)} \rho(v)\rho(v_*) \\ &\lesssim \left( \|f\|_{L^\infty(\mathbb{R}^3)} + \|g\|_{L^\infty(\mathbb{R}^3)} \right) \varepsilon^{-1} \left[ d_2^2(f,g) \right]^{1-\varepsilon}. \end{aligned}$$

Choosing $\varepsilon = \min\left( |\log d_2^2(f,g)|^{-1}, e^{-1} \right)$ gives

$$\frac{d}{dt} d_2^2(f,g) \lesssim \left( \|f\|_{L^\infty(\mathbb{R}^3)} + \|g\|_{L^\infty(\mathbb{R}^3)} \right) d_2^2(f,g) \Big( 1 + |\log d_2^2(f,g)| \Big).$$

# Sketch of the proof

Recreating the computations for the heat equation shows:

$$\frac{d}{dt} d_2^2(f,g) \lesssim \int_0^1 \int |v-v_*|^{-3} |\nabla u(v) - \nabla u(v_*)|^2 \rho(v)\rho(v_*)\, dv\, dv_*\, ds$$

We use

$$|\nabla u(v) - \nabla u(v_*)| \lesssim_s |v - v_*|, \quad 0 < s < 1.$$

Then, for $0 < \varepsilon < 1$

$$\begin{aligned} \frac{d}{dt} d_2^2(f,g) &\lesssim \int |v - v_*|^{-3+2\varepsilon} |\nabla u(v) - \nabla u(v_*)|^{2(1-\varepsilon)} \rho(v)\rho(v_*) \\ &\lesssim \left( \|f\|_{L^\infty(\mathbb{R}^3)} + \|g\|_{L^\infty(\mathbb{R}^3)} \right) \varepsilon^{-1} \left[ d_2^2(f,g) \right]^{1-\varepsilon}. \end{aligned}$$

Choosing $\varepsilon = \min\left( |\log d_2^2(f,g)|^{-1}, e^{-1} \right)$ gives

$$\frac{d}{dt} d_2^2(f,g) \lesssim \left( \|f\|_{L^\infty(\mathbb{R}^3)} + \|g\|_{L^\infty(\mathbb{R}^3)} \right) d_2^2(f,g) \Big( 1 + |\log d_2^2(f,g)| \Big).$$

# Sketch of the proof

Recreating the computations for the heat equation shows:

$$\frac{d}{dt}d_2^2(f,g) \lesssim \int_0^1 \int |v-v_*|^{-3}|\nabla u(v)-\nabla u(v_*)|^2\rho(v)\rho(v_*)\,dv\,dv_*\,ds$$

We use

$$|\nabla u(v)-\nabla u(v_*)| \lesssim_s |v-v_*|, \quad 0<s<1.$$

Then, for $0<\varepsilon<1$

$$\begin{aligned}\frac{d}{dt}d_2^2(f,g) &\lesssim \int |v-v_*|^{-3+2\varepsilon}\,|\nabla u(v)-\nabla u(v_*)|^{2(1-\varepsilon)}\,\rho(v)\rho(v_*)\\ &\lesssim \left(\|f\|_{L^\infty(\mathbb{R}^3)}+\|g\|_{L^\infty(\mathbb{R}^3)}\right)\varepsilon^{-1}\left[d_2^2(f,g)\right]^{1-\varepsilon}.\end{aligned}$$

Choosing $\varepsilon = \min\left(|\log d_2^2(f,g)|^{-1}, e^{-1}\right)$ gives

$$\frac{d}{dt}d_2^2(f,g) \lesssim \left(\|f\|_{L^\infty(\mathbb{R}^3)}+\|g\|_{L^\infty(\mathbb{R}^3)}\right)d_2^2(f,g)\Big(1+|\log d_2^2(f,g)|\Big).$$

# The fuzzy Landau equation

The lifting property also holds for the **fuzzy Landau equation**:

$$\partial_t f(x,v) + v \cdot \nabla_x f(x,v) = \tilde{q}(f)(x,v),$$

where

$$\tilde{q}(f) = \mathrm{div}_v \int_{\mathbb{R}^6} \kappa(x-x_*)\Phi(v-v_*)(\nabla_v - \nabla_{v_*})(f(x,v)f(x_*,v_*))\,dx_*\,dv_*,$$

and $\Phi(z) = |z|^{-1}\mathbb{P}(z)$, $\mathbb{P}(z) = \mathrm{Id}_3 - \frac{z \otimes z}{|z|^2}$.

- If $\kappa = \delta_0$, we recover the Landau equation.
- Intermediate model between homogeneous and inhomogeneous.
- Lifting property holds.

# The fuzzy Landau equation

The lifting property also holds for the **fuzzy Landau equation**:

$$\partial_t f(x,v) + v \cdot \nabla_x f(x,v) = \tilde{q}(f)(x,v),$$

where

$$\tilde{q}(f) = \mathrm{div}_v \int_{\mathbb{R}^6} \kappa(x - x_*) \Phi(v - v_*)(\nabla_v - \nabla_{v_*})(f(x,v) f(x_*, v_*))\, dx_*\, dv_*,$$

and $\Phi(z) = |z|^{-1}\mathbb{P}(z)$, $\mathbb{P}(z) = \mathrm{Id}_3 - \frac{z \otimes z}{|z|^2}$.

- If $\kappa = \delta_0$, we recover the Landau equation.
- Intermediate model between homogeneous and inhomogeneous.
- Lifting property holds.

# Goals

- Extend uniqueness result to the fuzzy Landau equation.
- Treat **weak solutions** using Fournier's approach.
- Generalize to analogous equations.

A weak solution of fuzzy Landau satisfies

$$\frac{d}{dt}\int \varphi f\,dx\,dv = \int \Big[-(v\cdot\nabla_x\varphi) + 2(\kappa b * f)\cdot\nabla_v\varphi + (\kappa\Phi * f) : D_v^2\varphi\Big] f\,dx\,dv$$

We consider the general family

$$\partial_t f_t(x) + \mathrm{div}\Big(c(x)f_t + (b(x)*f_t)f_t\Big) = D^2 : \Big((A(x)*f_t)f_t + Mf_t\Big),$$

where $D:A = \sum_{i,j}\partial_{x_ix_j}a^{ij}$.

# Goals

- Extend uniqueness result to the fuzzy Landau equation.
- Treat **weak solutions** using Fournier's approach.
- Generalize to analogous equations.

A weak solution of fuzzy Landau satisfies

$$\frac{d}{dt}\int \varphi f\,dx\,dv = \int \Big[-(v\cdot\nabla_x\varphi) + 2(\kappa b * f)\cdot\nabla_v\varphi + (\kappa\Phi * f) : D_v^2\varphi\Big] f\,dx\,dv$$

We consider the general family

$$\partial_t f_t(x) + \mathrm{div}\Big(c(x)f_t + (b(x)*f_t)f_t\Big) = D^2 : \Big((A(x)*f_t)f_t + Mf_t\Big),$$

where $D : A = \sum_{i,j}\partial_{x_i x_j} a^{ij}$.

# Strategy

**Key idea:** generate couplings $\Pi_t \in \Gamma(f_t, g_t)$ and estimate

$$d_2^2(f_t, g_t) \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} |x - y|^2 \, d\Pi_t(x, y).$$

We prescribe an equation for $\Pi_t$.

**Toy model:** Consider $\partial_t f_t(x) = \Delta f_t(x)$, $\partial_t g_t(y) = \Delta g_t(y)$ and

$$\partial_t \Pi_t = \Delta_{x,y} \Pi_t, \qquad \Pi_0 = \Pi_0^{\text{opt}}.$$

Then

$$\frac{d}{dt} \int |x - y|^2 \, d\Pi_t = \int \Delta_{x,y} \left( |x - y|^2 \right) \Pi_t = 4.$$

Need a different equation for $\Pi_t$!

# Strategy

**Key idea:** generate couplings $\Pi_t \in \Gamma(f_t, g_t)$ and estimate

$$d_2^2(f_t, g_t) \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} |x-y|^2 \, d\Pi_t(x,y).$$

We prescribe an equation for $\Pi_t$.

**Toy model:** Consider $\partial_t f_t(x) = \Delta f_t(x)$, $\partial_t g_t(y) = \Delta g_t(y)$ and

$$\partial_t \Pi_t = \Delta_{x,y} \Pi_t, \qquad \Pi_0 = \Pi_0^{\text{opt}}.$$

Then

$$\frac{d}{dt} \int |x-y|^2 \, d\Pi_t = \int \Delta_{x,y} \left( |x-y|^2 \right) \Pi_t = 4.$$

Need a different equation for $\Pi_t$!

# Strategy

**Toy model:** Consider $\partial_t f_t(x) = \Delta f_t(x)$, $\partial_t g_t(y) = \Delta g_t(y)$ and

$$\partial_t \Pi_t = D^2_{x,y} : \left( \begin{pmatrix} I & I \\ I & I \end{pmatrix} \Pi_t \right), \qquad \Pi_0 = \Pi_0^{\mathrm{opt}}.$$

Then

$$\frac{d}{dt} \int |x-y|^2 \, d\Pi_t = \int \begin{pmatrix} I & I \\ I & I \end{pmatrix} : D^2_{x,y}(|x-y|^2) \, d\Pi_t$$
$$= 2 \int \begin{pmatrix} I & I \\ I & I \end{pmatrix} : \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} d\Pi_t = 0.$$

Therefore

$$d_2^2(f_t, g_t) \le \int |x-y|^2 \, d\Pi_t = \int |x-y|^2 \, d\Pi_0 = d_2^2(f_0, g_0).$$

# Strategy

**Toy model:** Consider $\partial_t f_t(x) = \Delta f_t(x)$, $\partial_t g_t(y) = \Delta g_t(y)$ and

$$\partial_t \Pi_t = D^2_{x,y} : \left( \begin{pmatrix} I & I \\ I & I \end{pmatrix} \Pi_t \right), \qquad \Pi_0 = \Pi_0^{\text{opt}}.$$

Then

$$\frac{d}{dt} \int |x-y|^2 \, d\Pi_t = \int \begin{pmatrix} I & I \\ I & I \end{pmatrix} : D^2_{x,y}(|x-y|^2) \, d\Pi_t$$
$$= 2 \int \begin{pmatrix} I & I \\ I & I \end{pmatrix} : \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} d\Pi_t = 0.$$

Therefore

$$d_2^2(f_t, g_t) \le \int |x-y|^2 \, d\Pi_t = \int |x-y|^2 \, d\Pi_0 = d_2^2(f_0, g_0).$$

# Main computation

$$
\begin{aligned}
&\partial_t \Pi_t + \mathsf{div}_{x,y}\left( \left[ \begin{pmatrix} c(x) \\ c(y) \end{pmatrix} + \begin{pmatrix} b(x) \\ b(y) \end{pmatrix} * \Pi_t^{\mathsf{opt}} \right] \Pi_t \right) \\
&\quad = D^2_{x,y} : \left( \begin{pmatrix} \mathfrak{m}(x)\mathfrak{m}^T(x) & \mathfrak{m}(x)\mathfrak{m}^T(y) \\ \mathfrak{m}(y)\mathfrak{m}^T(x) & \mathfrak{m}(y)\mathfrak{m}^T(y) \end{pmatrix} \Pi_t \right) \\
&\quad + D^2_{x,y} : \left( \begin{pmatrix} \sigma(x)\sigma^T(x) & \sigma(x)\sigma^T(y) \\ \sigma(y)\sigma^T(x) & \sigma(y)\sigma^T(y) \end{pmatrix} * \Pi_t^{\mathsf{opt}} \quad \Pi_t \right),
\end{aligned}
$$

where $M(x) = \mathfrak{m}(x)\mathfrak{m}^T(x)$, $A(x) = \sigma(x)\sigma^T(x)$. Then

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |x-y|^2 \, d\Pi_t \\
&= \int \left[ (x-y)\cdot(c(x)-c(y)) + |m(x)-m(y)|^2 \right] d\Pi_t \\
&+ \iint \left[ (x-y)\cdot(b(x-x_*)-b(y-y_*)) + |\sigma(x)-\sigma(y)|^2 \right] d\Pi_t \, d\Pi_t^{\mathsf{opt}}.
\end{aligned}
$$

# Main computation

$$
\begin{aligned}
&\partial_t \Pi_t + \mathrm{div}_{x,y}\left( \left[ \begin{pmatrix} c(x) \\ c(y) \end{pmatrix} + \begin{pmatrix} b(x) \\ b(y) \end{pmatrix} * \Pi_t^{\mathrm{opt}} \right] \Pi_t \right) \\
&\quad = D^2_{x,y} : \left( \begin{pmatrix} \mathfrak{m}(x)\mathfrak{m}^T(x) & \mathfrak{m}(x)\mathfrak{m}^T(y) \\ \mathfrak{m}(y)\mathfrak{m}^T(x) & \mathfrak{m}(y)\mathfrak{m}^T(y) \end{pmatrix} \Pi_t \right) \\
&\quad + D^2_{x,y} : \left( \begin{pmatrix} \sigma(x)\sigma^T(x) & \sigma(x)\sigma^T(y) \\ \sigma(y)\sigma^T(x) & \sigma(y)\sigma^T(y) \end{pmatrix} * \Pi_t^{\mathrm{opt}} \quad \Pi_t \right),
\end{aligned}
$$

where $M(x) = \mathfrak{m}(x)\mathfrak{m}^T(x)$, $A(x) = \sigma(x)\sigma^T(x)$. Then

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |x-y|^2 \, d\Pi_t \\
&= \int \left[ (x-y)\cdot(c(x)-c(y)) + |m(x)-m(y)|^2 \right] \, d\Pi_t \\
&+ \iint \left[ (x-y)\cdot(b(x-x_*)-b(y-y_*)) + |\sigma(x)-\sigma(y)|^2 \right] \, d\Pi_t \, d\Pi_t^{\mathrm{opt}}.
\end{aligned}
$$

# Singularity of the coefficients

**Problem:** we have $x = (x^1, \dots, x^k, x') \in (\mathbb{R}^d)^k \times \mathbb{R}^{n-dk}$ and

$$|b(x - x_*)| \lesssim \sum_{i=1}^{k} |x^i - x^i_*|^{-d+1}, \quad |\sigma(x - x_*)| \lesssim \sum_{i=1}^{k} |x^i - x^i_*|^{-\frac{d}{2}+1},$$

$$\frac{|b(x - x_*) - b(y - y_*)|}{|x - y| + |x_* - y_*|} \lesssim \sum_{i=1}^{k} \left( |x^i - x^i_*|^{-d} + |y^i - y^i_*|^{-d} \right),$$

$$\frac{|\sigma(x - x_*) - \sigma(y - y_*)|^2}{|x - y|^2 + |x_* - y_*|^2} \lesssim \sum_{i=1}^{k} \left( |x^i - x^i_*|^{-d} + |y^i - y^i_*|^{-d} \right).$$

This creates singular integrals which diverge logarithmically:

$$\int_{|x - x_*| \geq \varepsilon} |x^i - x^i_*|^{-3} f^{(i)}_t(x_*) \lesssim (1 - \log \varepsilon) \|f^{(i)}_t\|_{L^\infty}.$$

**Solution:** Divide integrals into the regions

$$R_1 = \left\{ |x^i - x^i_*| \geq \varepsilon, |y^i - y^i_*| \geq \varepsilon, \forall i \right\}, \quad R_2 = (\mathbb{R}^n)^4 \setminus R_1.$$

# Singularity of the coefficients

**Problem:** we have $x = (x^1, \dots, x^k, x') \in (\mathbb{R}^d)^k \times \mathbb{R}^{n-dk}$ and

$$|b(x - x_*)| \lesssim \sum_{i=1}^{k} |x^i - x_*^i|^{-d+1}, \quad |\sigma(x - x_*)| \lesssim \sum_{i=1}^{k} |x^i - x_*^i|^{-\frac{d}{2}+1},$$

$$\frac{|b(x - x_*) - b(y - y_*)|}{|x - y| + |x_* - y_*|} \lesssim \sum_{i=1}^{k} \left( |x^i - x_*^i|^{-d} + |y^i - y_*^i|^{-d} \right),$$

$$\frac{|\sigma(x - x_*) - \sigma(y - y_*)|^2}{|x - y|^2 + |x_* - y_*|^2} \lesssim \sum_{i=1}^{k} \left( |x^i - x_*^i|^{-d} + |y^i - y_*^i|^{-d} \right).$$

This creates singular integrals which diverge logarithmically:

$$\int_{|x - x_*| \geq \varepsilon} |x^i - x_*^i|^{-3} f_t^{(i)}(x_*) \lesssim (1 - \log \varepsilon) \| f_t^{(i)} \|_{L^\infty}.$$

**Solution:** Divide integrals into the regions

$$R_1 = \left\{ |x^i - x_*^i| \geq \varepsilon, |y^i - y_*^i| \geq \varepsilon, \forall i \right\}, \quad R_2 = (\mathbb{R}^n)^4 \setminus R_1.$$

# Singularity of the coefficients

**Problem:** we have $x = (x^1, \ldots, x^k, x') \in (\mathbb{R}^d)^k \times \mathbb{R}^{n-dk}$ and

$$|b(x-x_*)| \lesssim \sum_{i=1}^{k} |x^i - x_*^i|^{-d+1}, \quad |\sigma(x-x_*)| \lesssim \sum_{i=1}^{k} |x^i - x_*^i|^{-\frac{d}{2}+1},$$

$$\frac{|b(x-x_*) - b(y-y_*)|}{|x-y| + |x_* - y_*|} \lesssim \sum_{i=1}^{k} \left( |x^i - x_*^i|^{-d} + |y^i - y_*^i|^{-d} \right),$$

$$\frac{|\sigma(x-x_*) - \sigma(y-y_*)|^2}{|x-y|^2 + |x_* - y_*|^2} \lesssim \sum_{i=1}^{k} \left( |x^i - x_*^i|^{-d} + |y^i - y_*^i|^{-d} \right).$$

This creates singular integrals which diverge logarithmically:

$$\int_{|x-x_*|\geq\varepsilon} |x^i - x_*^i|^{-3} f_t^{(i)}(x_*) \lesssim (1 - \log\varepsilon) \|f_t^{(i)}\|_{L^\infty}.$$

**Solution:** Divide integrals into the regions

$$R_1 = \left\{ |x^i - x_*^i| \geq \varepsilon, |y^i - y_*^i| \geq \varepsilon, \forall i \right\}, \quad R_2 = (\mathbb{R}^n)^4 \setminus R_1.$$

# Main computation

The previous computation yields

$$\frac{d}{dt}\int |x-y|^2\,d\Pi_t \lesssim \max_i(\|f_t^{(i)}\|_{L^\infty}, \|g_t^{(i)}\|_{L^\infty})\left[\varepsilon + |\log\varepsilon|\int |x-y|^2\,d\Pi_t\right].$$

Then we set $\varepsilon = \min\left(\int |x-y|^2\,d\Pi_t, e^{-1}\right)$ we obtain

$$\frac{d}{dt}\int |x-y|^2\,d\Pi_t \lesssim \max_i(\|f_t^{(i)}\|_{L^\infty}, \|g_t^{(i)}\|_{L^\infty})\Psi\left(\int |x-y|^2\,d\Pi_t\right),$$

with $\Psi(s) = s(1 + |\log s|)$.This shows the stability estimate

$$d_2^2(f_t, g_t) \leq \int |x-y|^2\,d\Pi_t \leq \omega\left(\int |x-y|\,d\Pi_0\right) = \omega\left(d_2^2(f_0, g_0)\right),$$

where $\omega$ depends on $\|f^{(i)}\|_{L^1_t L^\infty_{x^i}}$, $\|g^{(i)}\|_{L^1_t L^\infty_{x^i}}$.

# Main computation

The previous computation yields

$$\frac{d}{dt}\int |x-y|^2\,d\Pi_t \lesssim \max_i(\|f_t^{(i)}\|_{L^\infty}, \|g_t^{(i)}\|_{L^\infty})\left[\varepsilon + |\log\varepsilon| \int |x-y|^2\,d\Pi_t\right].$$

Then we set $\varepsilon = \min\left(\int |x-y|^2\,d\Pi_t, e^{-1}\right)$ we obtain

$$\frac{d}{dt}\int |x-y|^2\,d\Pi_t \lesssim \max_i(\|f_t^{(i)}\|_{L^\infty}, \|g_t^{(i)}\|_{L^\infty})\Psi\left(\int |x-y|^2\,d\Pi_t\right),$$

with $\Psi(s) = s(1+|\log s|)$.This shows the stability estimate

$$d_2^2(f_t, g_t) \le \int |x-y|^2\,d\Pi_t \le \omega\left(\int |x-y|\,d\Pi_0\right) = \omega\Big(d_2^2(f_0, g_0)\Big),$$

where $\omega$ depends on $\|f^{(i)}\|_{L_t^1 L_{x^i}^\infty}$, $\|g^{(i)}\|_{L_t^1 L_{x^i}^\infty}$.

# Main computation

The previous computation yields

$$\frac{d}{dt}\int |x-y|^2\,d\Pi_t \lesssim \max_i(\|f_t^{(i)}\|_{L^\infty}, \|g_t^{(i)}\|_{L^\infty})\left[\varepsilon + |\log\varepsilon|\int |x-y|^2\,d\Pi_t\right].$$

Then we set $\varepsilon = \min\left(\int |x-y|^2\,d\Pi_t, e^{-1}\right)$ we obtain

$$\frac{d}{dt}\int |x-y|^2\,d\Pi_t \lesssim \max_i(\|f_t^{(i)}\|_{L^\infty}, \|g_t^{(i)}\|_{L^\infty})\Psi\left(\int |x-y|^2\,d\Pi_t\right),$$

with $\Psi(s) = s(1+|\log s|)$.This shows the stability estimate

$$d_2^2(f_t, g_t) \le \int |x-y|^2\,d\Pi_t \le \omega\left(\int |x-y|\,d\Pi_0\right) = \omega\Big(d_2^2(f_0, g_0)\Big),$$

where $\omega$ depends on $\|f^{(i)}\|_{L^1_t L^\infty_{x^i}}$, $\|g^{(i)}\|_{L^1_t L^\infty_{x^i}}$.

# Gronwall-like inequality

In summary, we obtain a stability estimate for

$$\partial_t f_t(x) + \mathrm{div}\Big(c(x) f_t + (b(x) * f_t) f_t\Big) = D^2 : \Big((A(x) * f_t) f_t + M f_t\Big),$$

conditional on the $x^i$ marginals satisfying $f^{(i)}, g^{(i)} \in L^1_t L^\infty_{x^i}$.

## Theorem (General)

*There exists a modulus of continuity $\omega$ such that*

$$\sup_{0 \le t \le T} d_2^2(f_t, g_t) \le \omega\left(d_2^2(f_0, g_0)\right).$$

*$\omega$ depends on $\|f^{(i)}\|_{L^1_t L^\infty_{x^i}}$, $\|g^{(i)}\|_{L^1_t L^\infty_{x^i}}$ and $T$.*

Next, we see particular cases of the result.

# Fuzzy Landau equation

The result applies to the fuzzy Landau equation:

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = \tilde{q}(f)(t,x,v),$$

where

$$\tilde{q}(f) = \mathrm{div}_v \int_{\mathbb{R}^6} \kappa(x-x_*)\Phi(v-v_*)(\nabla_v - \nabla_{v_*})\big(f(x,v)f(x_*,v_*)\big)\,dx_*\,dv_*.$$

## Theorem

*Given two weak solutions, there is a modulus of continuity with*

$$\sup_{0\le t\le T} d_2^2(f_t,g_t) \le \omega\left(d_2^2(f_0,g_0)\right),$$

*where $\omega$ depends on $\|f\|_{L^1_t L^\infty_v L^1_x}$, $\|g\|_{L^1_t L^\infty_v L^1_x}$ and $T$.*

# Multiespecies hom. Landau

It further applies to

$$\partial_t f^{(i)}(t,v^i) = \sum_{j=1}^{N} q_{ij}(f^{(i)}, f^{(j)})(t,v^i),$$

where

$$q_{ij}(f,g) := \frac{c_{ij}}{m_i}\mathrm{div}_{v^i}\int_{\mathbb{R}^3}\Phi(v^i - v_*^j)\left(\frac{\nabla_{v^i}}{m_i} - \frac{\nabla_{v_*^j}}{m_j}\right)(f(t,v^i)g(t,v_*^j))\,dv_*^j,$$

## Theorem

*Given two weak solutions, there is a modulus of continuity with*

$$\sup_{0\le t\le T}\sum_{i=1}^{N} d_2^2(f_t^{(i)}, g_t^{(i)}) \le \omega\left(\sum_{i=1}^{N} d_2^2(f_0^{(i)}, g_0^{(i)})\right),$$

*where $\omega$ depends on $\|f^{(i)}\|_{L_t^1 L_v^\infty}$, $\|g^{(i)}\|_{L_t^1 L_v^\infty}$, and $T$.*

# Vlasov-Poisson equation

We also recover a known uniqueness result on

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + E(t,x) \cdot \nabla_v f(t,x,v) = 0,$$

where

$$E(x) := \pm \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x - x_*}{|x - x_*|^3} \rho(x_*)\, dx_*, \quad \rho(x) := \int_{\mathbb{R}^3} f(x,v)\, dv.$$

## Theorem (Loeper, 2006)

*There exists a modulus of continuity such that*

$$\sup_{0 \le t \le T} d_2^2(f_t, g_t) \le \omega\left(d_2^2(f_0, g_0)\right),$$

*where* $\omega$ *depends on* $\|\rho_f\|_{L^1_t L^\infty_x}$, $\|\rho_g\|_{L^1_t L^\infty_x}$ *and* $T$.

# Patlak-Keller-Segel

We also recover a known uniqueness result on

$$\begin{cases} \partial_t f(t,x) + \operatorname{div}\left(f(t,x)\nabla u(t,x)\right) = \Delta f(t,x), \\ -\Delta u(t,x) + \alpha u(t,x) = f(t,x). \end{cases}$$

Note that

$$\nabla u = \nabla \Phi * f,$$

where $\Phi$ is the fundamental solution of $-\Delta u + \alpha u$.

**Theorem (Carrillo et al., 2012)**

*There exists a modulus of continuity with*

$$\sup_{0\le t\le T} d_2^2(f_t, g_t) \le \omega\left(d_2^2(f_0, g_0)\right),$$

*where $\omega$ depends on $\|f\|_{L^1_t L^\infty_x}$, $\|g\|_{L^1_t L^\infty_x}$, and $T$.*

# 2D incompressible Euler

We also recover known uniqueness results on

$$\partial_t \omega(t,x) + \mathsf{div}(u(t,x)\omega(t,x)) = 0,$$

where $u$ is given by Biot-Savart law

$$u(t,x) = \int_{\mathbb{R}^2} K_{\mathsf{BS}}(x - x_*)\omega(t,x_*)\,dx_*, \quad K_{\mathsf{BS}}(x) = \frac{1}{2\pi|x|^2}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

## Theorem (Yudovich, 1963)

*Given two weak solutions* $\omega, \tilde{\omega} \geq 0$*, with* $\omega, \tilde{\omega} \in L^1 \cap L^\infty$ *and* $\int \omega = \int \tilde{\omega}$*, there is a modulus of continuity* $\theta$ *with*

$$\sup_{0 \leq t \leq T} d_2^2(\omega_t, \tilde{\omega}_t) \leq \theta\left(d_2^2(\omega_0, \tilde{\omega}_0)\right),$$

*where* $\theta$ *depends on* $\|\omega_0\|_{L^\infty}$*,* $\|\tilde{\omega}_0\|_{L^\infty}$ *and* $T$*.*

# Construction of the couplings

We let $\Pi_t = \text{Law}(X_t, Y_t)$, where $(X_t, Y_t)$ are solutions of

$$\begin{aligned}
X_t = X_0 &+ \int_0^t c(X_s)\, ds + \int_0^t \mathfrak{m}(X_s)\, dB_s \\
&+ \int_0^t \int_{\mathbb{R}^n} b(X_s - x_*) f_s(x_*)\, dx_*\, ds \\
&+ \sqrt{2} \int_0^t \int_{\mathbb{R}^n \times \mathbb{R}^n} \sigma(X_s - x_*)\, W(dx_*, dy_*, ds), \\
Y_t = Y_0 &+ \int_0^t c(Y_s)\, ds + \int_0^t \mathfrak{m}(Y_s)\, dB_s \\
&+ \int_0^t \int_{\mathbb{R}^n} b(Y_s - y_*) g_s(y_*)\, dy_*\, ds \\
&+ \sqrt{2} \int_0^t \int_{\mathbb{R}^n \times \mathbb{R}^n} \sigma(Y_s - y_*)\, W(dx_*, dy_*, ds),
\end{aligned}$$

where $W(dx, dy, dt) = (W^1, W^2, W^3)$ is a white noise on $\mathbb{R}^6 \times \mathbb{R}^6 \times [0, T]$ with covariance measure $d\Pi_t^{\text{opt}}(x, y)dt$.

# Construction of the couplings

The SDEs are shown to have solutions through Picard iterations:

$$\Phi : L^\infty\left([0,T];L^2(\Omega)\right) \longrightarrow L^\infty\left([0,T];L^2(\Omega)\right),$$

$$\Phi(X)_t := X_0 + \int_0^t c(X_s)\,ds + \int_0^t \int_{\mathbb{R}^d} b(X_s - x_*) f_s(x_*)\,dx_*\,ds$$
$$+ \int_0^t \mathfrak{m}(X_s)\,dB_s + \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \sigma(X_s - x_*)\,W(dx_*, dy_*, ds).$$

Using similar computations, we shown

$$\frac{d}{dt}\mathbb{E}\left[|\Phi(X)_t - \Phi(\tilde{X})_t|^2\right]$$
$$\lesssim \max_{1\le i\le k}(\|f_t^{(i)}\|_{L^\infty})\left[\Psi\left(\mathbb{E}\left[|\Phi(X)_t - \Phi(\tilde{X}_t)|^2\right]\right) + \Psi\left(\mathbb{E}\left[|X_t - \tilde{X}_t|^2\right]\right)\right],$$

which is enough to show convergence of the iteration.